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Lebesgue Sobolev orthogonality on the unit circle

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Abstract

This paper is devoted to the study of asymptotic properties of the orthogonal polynomials with respect to a Sobolev inner product

$$\langle f(z), g(z)\rangle_s = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\mu(\theta) + \sum_{k=1}^{p} \lambda_k \int_0^{2\pi} f^{(k)}(e^{i\theta})\overline{g^{(k)}(e^{i\theta})}\frac{d\theta}{2\pi}, \quad z = e^{i\theta},$$

with $d\mu(\theta)$ a finite positive Borel measure on $[0, 2\pi]$ with an infinite set as support verifying the Szegő condition, $\lambda_1 > 0$, $\lambda_k \geqslant 0$ $(k = 2, \ldots, p)$ and $d\theta/2\pi$ the normalized Lebesgue measure on $[0, 2\pi]$.

Our aim is to extend some previous results that we have obtained in [2, 3] when the measure μ belongs to the Bernstein-Szegő class and $p = 1$. © 1998 Elsevier Science B.V. All rights reserved.

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Let us consider a finite positive Borel measure μ on $[0, 2\pi]$ with an infinite set as support.

The orthonormal polynomials on the unit circle with respect to the measure μ $\{\varphi_n\}$ are defined by

$$\int_0^{2\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\,d\mu(\theta) = \delta_{n,m} \quad m, n = 0, 1, 2, \ldots,$$

where $\varphi_n(z) = k_n z^n + \textit{lower degree terms}$ and $k_n > 0$.

We denote by $\{\Phi_n\}$ the monic orthogonal polynomial sequence (MOPS) with respect to the measure μ defined by $\Phi_n = k_n^{-1}\varphi_n$. It is well-known that $\{k_n\}$ is an increasing sequence and the

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norms of the MOPS have the following extremal property:

$$\min_{P=z^n+\dots} \|P\|_\mu = \|\Phi_n\|_\mu.$$

If μ' is the Radon–Nikodym derivative of the measure μ with respect to the Lebesgue measure, we say the measure μ verifies the Szegő condition if $\log\mu' \in L^1[0,2\pi]$. Szegő's theory deals with this last case which can be characterized by the following equivalent conditions:

$$\log \mu' \in L^1[0,2\pi] \iff \sum_{n=1}^{\infty} |\varphi_n(0)|^2 < \infty \iff \lim_{n\to\infty} k_n = k < \infty \iff$$

$$\overline{\mathbb{P}} \neq L^2_\mu[0,2\pi] \iff \prod_{n=1}^{\infty}(1-|\Phi_n(0)|^2) > 0 \iff \sum_{n=1}^{\infty} |\Phi_n(0)|^2 < \infty,$$

where $\mathbb{P}$ denotes the set of complex algebraic polynomials.

Standard references for these basic properties are [6, 7, 13].

In what follows, we consider the following Sobolev inner product:

$$\langle f(z), g(z)\rangle_s = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\mu(\theta) + \sum_{k=1}^{p} \lambda_k \int_0^{2\pi} f^{(k)}(e^{i\theta})\overline{g^{(k)}(e^{i\theta})}\frac{d\theta}{2\pi}, \quad z = e^{i\theta},$$

where μ is a finite positive Borel measure on $[0,2\pi]$ with an infinite set as support, $\lambda_1 > 0$, $\lambda_k \geqslant 0$ $(k = 2,\dots,p)$ and $d\theta/2\pi$ is the normalized Lebesgue measure on $[0,2\pi]$. Without loss of generality, we assume that μ is a probability measure.

If we denote by $\{\tilde{\Phi}_n\}$ the MOPS with respect to this Sobolev product, that we write in a more simple form

$$\langle f, g\rangle_s = \langle f, g\rangle_\mu + \sum_{k=1}^{p} \lambda_k \langle f^{(k)}, g^{(k)}\rangle_\theta,$$

we obtain the following properties concerning the norms and the asymptotic behavior of the Sobolev orthogonal polynomials. These properties are similar to those obtained in [3] when $p = 1$ and μ is a Bernstein–Szegő measure (see [8]).

Theorem 1.

(i) $$\lim_{n\to\infty} \frac{\|\tilde{\Phi}_n\|_s^2}{n^{2p}} = \lambda_p.$$

(ii) *There exists $N \in \mathbb{N}$ such that $\{\|\tilde{\Phi}_n\|_s\}_{n\geqslant N}$ is an increasing sequence.*

Proof. (i) By using the extremal property of the MOPS $\{\tilde{\Phi}_n\}$ with respect to the Sobolev norm we obtain

$$1 + \sum_{k=1}^{p} \lambda_k \prod_{j=0}^{k-1}(n-j)^2 = \|z^n\|_s^2 \geqslant \|\tilde{\Phi}_n\|_s^2 \geqslant \lambda_p \prod_{j=0}^{p-1}(n-j)^2$$

which implies (i).

In order to prove statement (ii), take into account that for the values of n such that

$$\lambda_1[n^2-(n-1)^2]+\lambda_2(n-1)^2[n^2-(n-2)^2]+\cdots$$
$$+\lambda_p(n-1)^2\dots(n-p+1)^2[n^2-(n-p)^2]\geqslant 1,$$

it holds

$$\|\tilde{\Phi}_n\|_s^2\geqslant\sum_{k=1}^{p}\lambda_k\prod_{j=0}^{k-1}(n-j)^2\geqslant 1+\sum_{k=1}^{p}\lambda_k\prod_{j=0}^{k-1}(n-j-1)^2\geqslant\|\tilde{\Phi}_{n-1}\|_s^2. \qquad \square$$

Theorem 2. *It holds that*

$$\|\tilde{\Phi}_n-z^n\|_s^2\leqslant 1-\|\tilde{\Phi}_n\|_\mu^2.$$

Proof. First, we take into account the following inequalities:

$$1+\sum_{k=1}^{p}\lambda_k\prod_{j=0}^{k-1}(n-j)^2=\|z^n\|_s^2\geqslant\|\tilde{\Phi}_n\|_s^2=\|\tilde{\Phi}_n\|_\mu^2$$
$$+\sum_{k=1}^{p}\lambda_k\|\tilde{\Phi}_n^{(k)}\|_\theta^2\geqslant\|\tilde{\Phi}_n\|_\mu^2+\sum_{k=1}^{p}\lambda_k\prod_{j=0}^{k-1}(n-j)^2. \tag{1}$$

Using the orthogonality properties of $\tilde{\Phi}_n$, we have

$$\|\tilde{\Phi}_n-z^n\|_s^2=\|z^n\|_s^2-\|\tilde{\Phi}_n\|_s^2. \tag{2}$$

On the other hand, by using Eq. (1) we obtain

$$\|z^n\|_s^2-\|\tilde{\Phi}_n\|_s^2\leqslant 1+\sum_{k=1}^{p}\lambda_k\prod_{j=0}^{k-1}(n-j)^2$$
$$-\|\tilde{\Phi}_n\|_\mu^2-\sum_{k=1}^{p}\lambda_k\prod_{j=0}^{k-1}(n-j)^2=1-\|\tilde{\Phi}_n\|_\mu^2.$$

Then it is immediate to deduce the result.

Note that we also have obtained that $\|\tilde{\Phi}_n\|_\mu^2\leqslant 1$. $\square$

Lemma 1. *If* $\tilde{\Phi}_n(z)=z^n+\sum_{l=0}^{n-1}A_{n,l}z^l$, *then it holds that* $\forall\varepsilon>0$ $\exists N\in\mathbb{N}$ *such that for* $n\geqslant N$ $|A_{n,l}|<\varepsilon$, $l=N,\dots,n-1$.

Proof. Again, if we use the extremal property of the orthogonal polynomials we have

$$1+\sum_{k=1}^{p}\lambda_k\prod_{j=0}^{k-1}(n-j)^2=\|z^n\|_s^2 \geqslant \|\widetilde{\Phi}_n\|_s^2=\|\widetilde{\Phi}_n\|_\mu^2$$

$$+\lambda_1\left(n^2+\sum_{l=1}^{n-1}l^2|A_{n,l}|^2\right)+\cdots+\lambda_p\left(\prod_{j=0}^{p-1}(n-j)^2+\sum_{l=p}^{n-1}l^2(l-1)^2\ldots(l-p+1)^2|A_{n,l}|^2\right),$$

which implies

$$\lambda_1 l^2|A_{n,l}|^2 \leqslant 1 \quad (l=1,\ldots,n-1).$$

Therefore for $\varepsilon>0$ there exists a natural number N such that $(1/\lambda_1)^{1/2}1/N<\varepsilon$ and thus for $n\geqslant N$ $|A_{n,l}|<\varepsilon$ $(l=N,\ldots,n-1)$.

Note that we also have proved that $|A_{n,l}|\leqslant(1/\lambda_1)^{1/2}$ for every n and $l=1,\ldots,n-1$. □

Lemma 2. *If μ verifies the Szegő condition then the sequence $\{\widetilde{\Phi}_n(0)\}$ is bounded.*

Proof. Since the measure μ satisfies the Szegő condition, then the increasing sequence $k_n=1/\|\Phi_n\|_\mu$ verifies that $\lim_{n\to\infty}k_n=k<\infty$.

Thus if we write

$$\widetilde{\Phi}_n(z)=\Phi_n(z)+\sum_{l=0}^{n-1}B_{n,l}\Phi_l(z),$$

we get $B_{n,0}=0$ and

$$\|\widetilde{\Phi}_n\|_\mu^2=\|\Phi_n\|_\mu^2+\sum_{l=1}^{n-1}|B_{n,l}|^2\|\Phi_l\|_\mu^2.$$

Since $k^{-2}\leqslant\|\Phi_n\|_\mu^2$ $\forall n$ and $\|\widetilde{\Phi}_n\|_\mu^2\leqslant 1$ $\forall n$, then $k^{-2}(1+\sum_{l=1}^{n-1}|B_{n,l}|^2)\leqslant 1$.

Therefore,

$$|\widetilde{\Phi}_n(0)|=|\Phi_n(0)+\sum_{l=1}^{n-1}B_{n,l}\Phi_l(0)|\leqslant\left(\sum_{l=1}^{n}|\Phi_l(0)|^2\right)^{1/2}\left(1+\sum_{l=1}^{n-1}|B_{n,l}|^2\right)^{1/2}\leqslant S^{1/2}k,$$

where we denote by $S=\sum_{n=0}^{\infty}|\Phi_n(0)|^2$. □

Theorem 3. *Let μ be a measure verifying the Szegő condition. If we denote the moments for the measure μ by $c_n=\langle z^n,1\rangle_\mu$, and we assume that $\sum_{n=0}^{\infty}|c_n|<\infty$, then*

(i) $\lim_{n\to\infty}\|\widetilde{\Phi}_n-z^n\|_s=0$.

(ii) $\lim_{n\to\infty}\|\widetilde{\Phi}_n\|_\mu=1$.

Proof. (i) Since $\widetilde{\Phi}_n = z^n + \sum_{l=0}^{n-1} A_{n,l} z^l$, then

$$\|\widetilde{\Phi}_n\|_s^2 = \langle \widetilde{\Phi}_n(z), z^n \rangle_s = 1 + \sum_{l=0}^{n-1} A_{n,l} c_{l-n} + \sum_{k=1}^{p} \lambda_k \prod_{j=0}^{k-1} (n-j)^2 \tag{3}$$

and

$$\|\widetilde{\Phi}_n\|_s^2 = \|\widetilde{\Phi}_n\|_\mu^2 + \lambda_1 \left(n^2 + \sum_{l=1}^{n-1} l^2 |A_{n,l}|^2 \right) + \cdots$$
$$+ \lambda_p \left(\prod_{j=0}^{p-1} (n-j)^2 + \sum_{l=p}^{n-1} l^2 (l-1)^2 \ldots (l-p+1)^2 |A_{n,l}|^2 \right). \tag{4}$$

From (3) and (4) we get

$$1 + \sum_{l=0}^{n-1} A_{n,l} c_{l-n} = \|\widetilde{\Phi}_n\|_\mu^2 + \lambda_1 \sum_{l=1}^{n-1} l^2 |A_{n,l}|^2 + \cdots$$
$$+ \lambda_p \sum_{l=p}^{n-1} l^2 (l-1)^2 \ldots (l-p+1)^2 |A_{n,l}|^2. \tag{5}$$

Next, we prove that $\lim_{n\to\infty} \sum_{l=0}^{n-1} A_{n,l} \overline{c_{n-l}} = 0$.

Given $\varepsilon' > 0$, let us consider $\varepsilon > 0$ such that $\varepsilon \leqslant \varepsilon'/2C$ with $C = \sum_{n=0}^{\infty} |c_n|$, (for this last condition see [1]). From Lemma 1 we have that $\exists N : \forall n \geqslant N \quad |A_{n,l}| < \varepsilon \quad l = N, \ldots, n-1$ and, therefore,

$$\left| \sum_{l=N}^{n-1} A_{n,l} \overline{c_{n-l}} \right| < \varepsilon \sum_{j=1}^{n-N} |c_j| \leqslant \varepsilon C \leqslant \frac{\varepsilon'}{2}.$$

Besides, from Lemmas 1 and 2, we get $|A_{n,l}| \leqslant \max\{(\frac{1}{\lambda_1})^{1/2}, S^{1/2}k\} = M$ for $l = 0, \ldots, n-1$.

On the other hand, since $\lim_{n\to\infty} c_n = 0$, there exists m such that $\forall n \geqslant m \quad |c_n| < \varepsilon'/2MN$. Then if we consider $n \geqslant m + N - 1$ it holds that

$$\left| \sum_{l=0}^{N-1} A_{n,l} \overline{c_{n-l}} \right| < \frac{\varepsilon'}{2MN} \sum_{l=0}^{N-1} |A_{n,l}| \leqslant \frac{\varepsilon'}{2MN} MN = \frac{\varepsilon'}{2}.$$

Whence, given $\varepsilon' > 0$ there exists $p = m + N - 1$ such that $\forall n \geqslant p \quad |\sum_{l=0}^{n-1} A_{n,l} \overline{c_{n-l}}| < \varepsilon'$.

Then, from (5)

$$\lim_{n\to\infty} \left(\|\widetilde{\Phi}_n\|_\mu^2 + \lambda_1 \sum_{l=1}^{n-1} l^2 |A_{n,l}|^2 + \ldots + \lambda_p \sum_{l=p}^{n-1} l^2 (l-1)^2 \ldots (l-p+1)^2 |A_{n,l}|^2 \right) = 1.$$

Taking into account (4) we conclude

$$\lim_{n\to\infty} \left(\|\widetilde{\Phi}_n\|_s^2 - \sum_{k=1}^{p} \lambda_k \prod_{j=0}^{k-1} (n-j)^2 \right) = 1.$$

Finally, since $\|z^n\|_s^2 = 1 + \sum_{k=1}^p \lambda_k \prod_{j=0}^{k-1}(n-j)^2$, then $\lim_{n\to\infty}(\|\widetilde{\Phi}_n\|_s^2 + 1 - \|z^n\|_s^2) = 1$ and, therefore, $\lim_{n\to\infty}(\|\widetilde{\Phi}_n\|_s^2 - \|z^n\|_s^2) = 0$. Applying (2) we get the result.

We note that from our result, it is immediate to deduce that the following limits are also zero:

$$\lim_{n\to\infty} \|\widetilde{\Phi}_n - z^n\|_\mu = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\widetilde{\Phi}'_n - nz^{n-1}\|_\theta = 0.$$

(ii) Since $\widetilde{\Phi}_n(z) = z^n + \widetilde{\Phi}_n(z) - z^n$, we have

$$1 - \|\widetilde{\Phi}_n - z^n\|_\mu = \|z^n\|_\mu - \|\widetilde{\Phi}_n - z^n\|_\mu \leqslant \|\widetilde{\Phi}_n\|_\mu \leqslant 1,$$

where the last inequality follows from Theorem 2.

Then, taking limits and applying (i) we obtain the result. □

Theorem 4. *If μ verifies the Szegő condition then*

$$\lim_{n\to\infty} \frac{\widetilde{\Phi}_n(z)}{z^n} = 1 \quad \textit{uniformly on compact subsets of } |z| > 1.$$

Proof. Let z such that $1 < r \leqslant |z| \leqslant R$. For a fixed $\varepsilon' > 0$, take $\varepsilon > 0$ such that $\varepsilon/(r-1) < \varepsilon'/2$.

Using Lemma 1 we get that for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that, for $n \geqslant N$, $|A_{n,l}| < \varepsilon$, $l = N, \ldots, n-1$.

Therefore,

$$\left|\sum_{l=N}^{n-1} \frac{A_{n,l}}{z^{n-l}}\right| < \varepsilon \sum_{l=1}^{n-N} \frac{1}{|z|^l} < \frac{\varepsilon}{|z|-1} < \frac{\varepsilon}{r-1} < \frac{\varepsilon'}{2}.$$

On the other hand, for $\varepsilon'/2N > 0$ there exists $m \in \mathbb{N}$ such that, for $n \geqslant m$, $M/r^n < \varepsilon'/2N$, with M defined in the proof of the preceding theorem. Whence for $n \geqslant m + N - 1$ we have

$$\left|\sum_{l=0}^{N-1} \frac{A_{n,l}}{z^{n-l}}\right| \leqslant M \sum_{l=n-N+1}^{n} \frac{1}{r^l} < N\frac{\varepsilon'}{2N} = \frac{\varepsilon'}{2}.$$

Then, for $\varepsilon' > 0$ there exists $p = m + N - 1$ such that, for $n \geqslant p$,

$$\left|\frac{\widetilde{\Phi}_n(z)}{z^n} - 1\right| < \varepsilon', \; 1 < r \leqslant |z| \leqslant R,$$

which yields the result. □

By applying the following result, due to Cauchy (see [10]): "The zeros of $\widetilde{\Phi}_n$ are in the annulus $|A_{n,0}|/(|A_{n,0}| + L') < |z| < 1 + L$ with $L = \max_{l=1,\ldots,n} |A_{n,n-l}|$ and $L' = \max_{l=1,\ldots,n-1}\{1, |A_{n,n-l}|\}$", we get:

Theorem 5. *If μ verifies the Szegő condition, then the zeros of $\widetilde{\Phi}_n(z)$ are in the disk $|z| < 1 + \max\{kS^{1/2}, (\frac{1}{\lambda_1})^{1/2}\}$.*

Proof. From Lemmas 1 and 2, $|A_{n,0}| \leqslant S^{1/2}k$ and $|A_{n,n-l}| \leqslant (\frac{1}{\lambda_1})^{1/2}$ for $l = 1, \dots, n-1$. Therefore, $L' \leqslant \max\{1, (\frac{1}{\lambda_1})^{1/2}\}$ and $L \leqslant \max\{(\frac{1}{\lambda_1})^{1/2}, S^{1/2}k\}$. □

Corollary 1. *If μ verifies the Szegő condition then*

$$\lim_{n\to\infty} \frac{\tilde{\Phi}_n(z)}{\Phi_n(z)} = k\overline{D\left(\mu', \frac{1}{\bar{z}}\right)} \quad \textit{uniformly on compact subsets of } |z| > 1.$$

Proof. In our conditions it is known that

$$\lim_{n\to\infty} \frac{\Phi_n(z)}{\|\Phi_n\|_\mu z^n} = \frac{1}{\overline{D(\mu', \frac{1}{\bar{z}})}} \qquad |z| > 1$$

and this holds uniformly for $|z| \geqslant r > 1$, (see [12, 13]). We recall that $D(\mu', z)$ is the Szegő function of μ' defined by

$$D(\mu', z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \mu'(\theta)\, d\theta\right) \quad \text{for} \quad |z| < 1.$$

Therefore, if we apply Theorem 4 we get

$$\lim_{n\to\infty} \frac{\tilde{\Phi}_n(z)}{\Phi_n(z)} = \lim_{n\to\infty} \frac{\tilde{\Phi}_n(z)}{z^n} \frac{\|\Phi_n\|_\mu z^n}{\Phi_n(z)} \frac{1}{\|\Phi_n\|_\mu} = k\overline{D\left(\mu', \frac{1}{\bar{z}}\right)}. \qquad \square$$

Corollary 2. *Let $\varepsilon > 0$. If μ verifies the Szegő condition, then for n large enough the zeros of $\tilde{\Phi}_n(z)$ are in $|z| < 1 + \varepsilon$.*

Proof. The result follows immediately if we take into account Theorem 5 and apply Hurwitz's theorem, (see [4]), in Theorem 4. □

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